

SPECIAL CUBIC CREMONA TRANSFORMATIONS OF  $\mathbb{P}^6$  AND  $\mathbb{P}^7$ 

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**ABSTRACT.** A famous result of B. Crauder and S. Katz (1989) concerns the classification of special Cremona transformations whose base locus has dimension at most two. Furthermore, they also proved that a special Cremona transformation with base locus of dimension three has to be one of the following: 1) a quinto-quintic transformation of  $\mathbb{P}^5$ ; 2) a cubo-quintic transformation of  $\mathbb{P}^6$ ; or 3) a quadro-quintic transformation of  $\mathbb{P}^8$ . Special Cremona transformations as in case 1) have been classified by L. Ein and N. Shepherd-Barron (1989), while in our previous work (2013), we classified special quadro-quintic Cremona transformations of  $\mathbb{P}^8$ . The main aim here is to consider the problem of classifying special cubo-quintic Cremona transformations of  $\mathbb{P}^6$ , concluding the classification of special Cremona transformations whose base locus has dimension three.

## INTRODUCTION

In this paper, we investigate special Cremona transformations, i.e. birational transformations  $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$  of a complex projective space into itself, whose base locus  $\mathfrak{B} \subset \mathbb{P}^n$  is a smooth and irreducible variety. One says that  $\varphi$  is of type  $(\delta_1, \delta_2)$  if the linear system defining it (resp. defining its inverse) belongs to  $|\mathcal{O}_{\mathbb{P}^n}(\delta_1)|$  (resp.  $|\mathcal{O}_{\mathbb{P}^n}(\delta_2)|$ ). When the dimension of the base locus is at most two, B. Crauder and S. Katz obtained the following results:

**Theorem 0.1** ([CK89], see also [Kat87]). *Let  $\varphi$  be a special Cremona transformation of  $\mathbb{P}^n$  whose base locus  $\mathfrak{B}$  has dimension 1. Then one of the two following cases occurs:*

- (1)  $n = 3$ ,  $\varphi$  is of type  $(3, 3)$ , and  $\mathfrak{B}$  is a curve of degree 6 and genus 3;
- (2)  $n = 4$ ,  $\varphi$  is of type  $(2, 3)$ , and  $\mathfrak{B}$  is a curve of degree 5 and genus 1.

In the same paper, they also proved the following:

**Theorem 0.2** ([CK89], see also [ST69, ST70]). *Let  $\varphi$  be a special Cremona transformation of  $\mathbb{P}^n$  whose base locus  $\mathfrak{B}$  has dimension 2. Then one of the following cases occurs:*

- (1)  $n = 4$ ,  $\varphi$  is of type  $(3, 2)$ , and  $\mathfrak{B}$  is a quintic elliptic scroll;
- (2)  $n = 4$ ,  $\varphi$  is of type  $(4, 4)$ , and  $\mathfrak{B}$  is a degree 10 determinantal surface given by the  $4 \times 4$  minors of a  $4 \times 5$  matrix of linear forms;
- (3)  $n = 5$ ,  $\varphi$  is of type  $(2, 2)$ , and  $\mathfrak{B}$  is the Veronese surface;
- (4)  $n = 6$ ,  $\varphi$  is of type  $(2, 4)$ , and  $\mathfrak{B}$  is a septic elliptic scroll;
- (5)  $n = 6$ ,  $\varphi$  is of type  $(2, 4)$ , and  $\mathfrak{B}$  is  $\mathbb{P}^2$  blown up at eight points and embedded in  $\mathbb{P}^6$  as an octic surface by quartic curves passing through all eight points.

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Furthermore, when  $\mathfrak{B}$  has dimension 3, we have the following possible cases (see [CK91, Corollary 1], and [ESB89, Theorem 3.2] for the second part of the statement in (1) below):

**Proposition 0.3.** *Let  $\varphi$  be a special Cremona transformation of  $\mathbb{P}^n$  whose base locus  $\mathfrak{B}$  has dimension 3. Then one of the following cases occurs:*

- (1)  $n = 5$ ,  $\varphi$  is of type  $(5, 5)$ , and  $\mathfrak{B}$  is a degree 15 determinantal threefold given by the  $5 \times 5$  minors of a  $5 \times 6$  matrix of linear forms;
- (2)  $n = 6$  and  $\varphi$  is of type  $(3, 5)$ ;
- (3)  $n = 8$  and  $\varphi$  is of type  $(2, 5)$ .

In [Sta13], we classified special Cremona transformations as in case (3) above, by showing the following:

**Theorem 0.4** ([Sta13]). *Let  $\varphi$  be a special quadratic Cremona transformation of  $\mathbb{P}^n$  whose base locus  $\mathfrak{B}$  has dimension 3. Then  $n = 8$ ,  $\varphi$  is of type  $(2, 5)$ , and one of the two following cases occurs:*

- (1)  $\mathfrak{B}$  is a scroll over a rational surface  $Y$  with  $K_Y^2 = 5$ , and it is embedded in  $\mathbb{P}^8$  as a threefold of degree 12 and sectional genus 6;
- (2)  $\mathfrak{B}$  is the blow-up of a Fano variety at one point, embedded in  $\mathbb{P}^8$  as a threefold of degree 13 and sectional genus 8.

In [Sta15], we exhibited an example of Cremona transformation as in case (1) above, while examples of transformations as in case (2) had been constructed in [HKS92]. The main aim of this paper is to prove Theorem 0.5 below, which describes the transformations as in case (2) of Proposition 0.3, concluding the classification of special Cremona transformations whose base locus has dimension three.

**Theorem 0.5.** *Let  $\varphi$  be a special cubic Cremona transformation of  $\mathbb{P}^n$  whose base locus  $\mathfrak{B}$  has dimension 3. Then  $n = 6$ ,  $\varphi$  is of type  $(3, 5)$ , and one of the two following cases occurs:*

- (A)  $\mathfrak{B}$  is a threefold of degree 14, sectional genus 15, and it is Pfaffian (i.e. given by the Pfaffians of a skew-symmetric matrix) with trivial canonical bundle;
- (B)  $\mathfrak{B}$  is a conic bundle over  $\mathbb{P}^2$ , embedded in  $\mathbb{P}^6$  as a threefold of degree 13 and sectional genus 12.

An example as in the first case of Theorem 0.5 can be obtained by taking the Pfaffians of the principal  $6 \times 6$  minors of a general  $7 \times 7$  skew-symmetric matrix of linear forms on  $\mathbb{P}^6$ , while an example as in the second case is a bit more complicated; see Section 4.

From another point of view, if  $\varphi$  is a special Cremona transformation of  $\mathbb{P}^n$  and  $n \leq 6$ , then the dimension of its base locus is at most three (see [ESB89, Theorem 3.2]). Thus Theorem 0.5 also concludes the classification of the special Cremona transformations of  $\mathbb{P}^n$  with  $n \leq 6$ . We know no examples of special Cremona transformations of  $\mathbb{P}^7$ , but the second main result of this paper is the following:

**Theorem 0.6.** *Let  $\varphi$  be a special Cremona transformation of  $\mathbb{P}^7$ . Then  $\varphi$  is of type  $(3, 3)$  and its base locus is a fourfold of degree 12 and sectional genus 10, which is a fibration over  $\mathbb{P}^1$  whose generic fibre is a sextic del Pezzo threefold.*

For the convenience of the reader, in Table 1 below we summarize the classification of the special Cremona transformations of  $\mathbb{P}^n$  with either  $n \leq 7$  or dimension of the base locus  $r \leq 3$ .

	$r$	$n$	Projective degrees	Description of the base locus
I	1	3	3, 3	determinantal curve
II		4	2, 4, 3	quintic elliptic curve
III	2	4	3, 4, 2	quintic elliptic scroll in lines
IV		4	4, 6, 4	determinantal surface
V		5	2, 4, 4, 2	Veronese surface
VI		6	2, 4, 8, 9, 4	septic elliptic scroll in lines
VII		6	2, 4, 8, 8, 4	rational octic surface
VIII	3	5	5, 10, 10, 5	determinantal threefold
IX		8	2, 4, 8, 16, 20, 14, 5	scroll over rational surface with $K^2 = 5$
X		8	2, 4, 8, 16, 19, 13, 5	Fano threefold with one point blown up
XI		6	3, 9, 13, 11, 5	Pfaffian threefold
XII		6	3, 9, 14, 12, 5	conic bundle over a plane
XIII	4	7	3, 9, 15, 15, 9, 3	del Pezzo fibration over a line

TABLE 1. Special Cremona transformations of  $\mathbb{P}^n$  with either  $n \leq 7$  or  $r \leq 3$ .

The outline of the paper is as follows. In Section 1, we give some background information; in particular, we recall some adjunction-theoretic results.

In Section 2, we prove Theorem 0.5, and we will proceed essentially in four steps. Firstly, we compute the numerical invariants of  $\varphi$  and  $\mathfrak{B}$  as functions of  $\lambda = \deg(\mathfrak{B})$  and of the sectional genus  $g$  of  $\mathfrak{B}$ . In particular, we determine the Hilbert polynomial of  $\mathfrak{B}$  (Lemma 2.2), the degrees of the Chern and Segre classes of the tangent and normal bundle of  $\mathfrak{B}$  (Lemma 2.3), and the projective degrees of  $\varphi$  (Lemma 2.4). Secondly, applying general inequalities for the invariants of threefolds and for the projective degrees of Cremona transformations, we reduce drastically the number of pairs  $(\lambda, g)$  for which such a transformation may exist (see Lemma 2.5). Thirdly, we apply adjunction theory to deduce a first rough result (Proposition 2.8): either  $(\mathfrak{B}, H_{\mathfrak{B}})$  is minimal of log-general type, or it admits a reduction to a conic bundle over a surface. Four and last, we refine our result by applying to the general hyperplane section of  $\mathfrak{B}$  a formula, due to P. Le Barz, concerning the number of 4-secants of surfaces in  $\mathbb{P}^5$ .

In Section 3, we prove Theorem 0.6, basically by altering slightly the proof of Theorem 0.5 in order to study the restriction of a special Cremona transformation of  $\mathbb{P}^7$  to a general hyperplane  $\mathbb{P}^6 \subset \mathbb{P}^7$ ; indeed one has that this restriction is a cubic transformation of  $\mathbb{P}^6$  (into a hypersurface of  $\mathbb{P}^7$ ) with a smooth irreducible threefold as base locus.

In Section 4, among others examples, we explain how to construct an example of Cremona transformation as in case (B) of Theorem 0.5. For this, we will use MACAULAY2 [GS16] with the *Cremona* package [Sta17].

We point out that some proofs in this paper are reduced to purely mechanical procedures of finding out the elements of a given finite set of tuples of integers that satisfy certain given relations. So a computer is very helpful, although not indispensable.

## 1. PRELIMINARIES

**1.1. Overview on adjunction theory for threefolds.** Since some adjunction theoretic results are central for us, in this subsection we summarize them for the convenience of the reader. For details, proofs, and history on these results, we refer to the papers [Fuj90, BS95, Ion84, Ion86,

Som86, SV87, BBS89] and references therein. As usual, we do not distinguish between line bundles and divisors.

Let  $X$  be a smooth irreducible threefold,  $H$  a very ample divisor on  $X$ , and denote by  $K$  the canonical divisor of  $X$ . The degree of the polarized pair  $(X, H)$  is  $H^3$ , and the sectional genus  $g = g(X, H)$  is defined by  $2g - 2 = (K + 2H)H^2$ .

A pair  $(X', H')$ , where  $X'$  is a smooth irreducible threefold and  $H'$  an ample divisor on  $X'$ , is said to be a *reduction* of  $(X, H)$  if there is a morphism  $\pi : X \rightarrow X'$  expressing  $X$  as the blowing-up of  $X'$  at a finite number of distinct points  $p_1, \dots, p_\nu$  (not infinitely near), and moreover, denoting by  $E_i = \pi^{-1}(p_i)$  the exceptional divisors and by  $K'$  the canonical divisor of  $X'$ , we have  $H \approx \pi^*H' - \sum_{i=1}^\nu E_i$  or equivalently  $K + 2H \approx \pi^*(K' + 2H')$ . Of course, we have  $H'^3 = H^3 + \nu$  and  $g(X', H') = g(X, H)$ . When  $K + 2H$  is nef and big, there is a reduction  $(X', H')$ , uniquely determined by  $(X, H)$ , such that  $K' + 2H'$  is ample. So, in this case, we refer to  $(X', H')$  as *the (minimal) reduction* of  $(X, H)$ , and denote by  $\nu = \nu(X, H)$  the number of points blown up on  $X'$ .

Consider the complete linear system  $|K + 2H|$  on  $X$ . It is base-point free unless  $(X, H)$  is one of the following:  $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$ , the quadric  $(Q^3, \mathcal{O}_{Q^3}(1))$ , or a scroll over a curve. The map  $\Phi = \Phi_{|K+2H|}$ , when defined, is called *the adjunction map*, and we write  $\Phi = \sigma \circ \hat{\Phi}$  for the Remmert-Stein factorization of  $\Phi$ ; so  $\hat{\Phi}$  is a morphism with connected fibers onto a normal variety, and  $\sigma$  is a finite map.

**Fact 1.1.** *If the adjunction map  $\Phi$  is defined, then there are the following possibilities:*

- (I)  $\dim \Phi(X) = 0$  and  $(X, H)$  is a del Pezzo variety, i.e. a Fano variety of coindex 2;
- (II)  $\dim \Phi(X) = 1$  and  $\hat{\Phi}$  expresses  $X$  as a quadric fibration over a curve (i.e. any closed fibre of  $\hat{\Phi}$  is embedded as a quadric surface by the restriction of  $H$ );
- (III)  $\dim \Phi(X) = 2$  and  $\hat{\Phi}$  expresses  $X$  as a scroll over a smooth surface (i.e. any fibre of  $\hat{\Phi}$  is embedded as a line by the restriction of  $H$ );
- (IV)  $\dim \Phi(X) = 3$ ,  $X' := \hat{\Phi}(X)$  is smooth,  $H' := \hat{\Phi}_*(H)$  is ample, and  $(X', H')$  is the minimal reduction of  $(X, H)$ ; moreover  $K' + 2H'$  is very ample ( $K'$  stands for the canonical divisor of  $X'$ ).

Thus, except for a small list of well understood cases,  $K + 2H$  is nef and big, and  $(X, H)$  admits a reduction  $(X', H')$  with  $K' + 2H'$  ample. When moreover  $K' + H'$  is nef and big, one says that  $(X', H')$  is of *log-general type*. In this case, a smooth member of  $|H'|$  is a minimal surface of general type. We have the following:

**Fact 1.2.** *Let  $(X', H')$  be a reduction of  $(X, H)$ . Then  $K' + H'$  is nef and big unless either:*

- (I)  $(X', H') \simeq (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$ ;
- (II)  $(X', H') \simeq (Q, \mathcal{O}_Q(2))$ , where  $Q$  is a quadric in  $\mathbb{P}^4$ ;
- (III)  $(X', H')$  is a Veronese fibration over a smooth curve  $Y$ , hence there is a surjective morphism  $\pi : X' \rightarrow Y$  whose general fibre is  $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$  and  $2K' + 3H' \approx \pi^*\mathcal{L}$  for some ample line bundle  $\mathcal{L}$  on  $Y$ ;
- (IV)  $(X', H')$  is a Mukai variety, i.e. a Fano variety of coindex 3;
- (V)  $(X', H')$  is a del Pezzo fibration over a smooth curve  $Y$ , hence there exists a surjective morphism  $\pi : X' \rightarrow Y$  and  $K' + H' \approx \pi^*\mathcal{L}$  for some ample line bundle  $\mathcal{L}$  on  $Y$ ;
- (VI)  $(X', H')$  is a conic bundle over a surface  $Y$ , hence there exists a surjective morphism  $\pi : X' \rightarrow Y$  and  $K' + H' \approx \pi^*\mathcal{L}$  for some ample line bundle  $\mathcal{L}$  on  $Y$ .

The *pluridegrees* of  $(X', H')$ , for  $j = 0, \dots, 3$ , where  $(X', H')$  is the reduction of  $(X, H)$ , are defined as  $d_j = d_j(X', H') = (K' + H')^j H'^{3-j}$ . The following result collects several inequalities, proved in [BBS89], for the pluridegrees in the case when  $(X', H')$  is of log-general type.

**Fact 1.3.** *Assume that  $K + 2H$  is nef and big, and let  $(X', H')$  be the reduction of  $(X, H)$ . If  $K' + H'$  is nef and big, then the following inequalities hold:*

- (i)  $d_1 \geq 1, d_2 \geq 3, d_3 \geq 1$ ;
- (ii)  $d_1^2 \geq d_2 d_0, d_2^2 \geq d_3 d_1$ ;
- (iii)  $d_1^3 \geq d_3 d_0^2, d_2^3 \geq d_3^2 d_0$ ;
- (iv)  $5d_1 \geq d_0, 5d_2 \geq d_1, 5d_3 \geq d_2$ ;
- (v) if it is not true that  $d_3 = 1, d_2 = 5$ , and  $d_1 \leq 25$ , then  $4d_1 \geq d_0, 4d_2 \geq d_1$ , and  $4d_3 \geq d_2$ ;
- (vi)  $2(\chi(\mathcal{O}_{X'}) - \chi(\mathcal{O}_{X'}(-H'))) - 6 \leq d_2 < 9(\chi(\mathcal{O}_{X'}) - \chi(\mathcal{O}_{X'}(-H')))$ ;
- (vii)  $(3d_2 + 2d_1 - d_0 + 12(\chi(\mathcal{O}_{X'}) - \chi(\mathcal{O}_{X'}(-H')))) / 32 \geq \chi(\mathcal{O}_{X'})$ ;
- (viii)  $2d_3 + 7d_2 + 12d_1 - 3d_0 + 30\chi(\mathcal{O}_{X'}) + 18\chi(\mathcal{O}_{X'}(-H')) \geq 0$ ;
- (ix)  $24(\chi(\mathcal{O}_{X'}) - \chi(\mathcal{O}_{X'}(-H'))) + 2g - 2 - 2d_2 - c_3(\mathcal{S}_{X'}) \geq 0$ .

**1.2. Projective degrees of Cremona transformations.** Recall that the  $k$ -th projective degree  $\deg_k(\varphi)$  of a Cremona transformation  $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$  is the degree of the inverse image  $\varphi^{-1}(\mathbb{P}^{n-k})$  of a general linear subspace  $\mathbb{P}^{n-k} \subseteq \mathbb{P}^n$ ; in other words, the projective degrees of  $\varphi$  are the multidegree of its graph, considered as a cycle on  $\mathbb{P}^n \times \mathbb{P}^n$ . The following general result puts restrictions on the possible projective degrees of Cremona transformations. The first part is easy and due to L. Cremona; the second part is an application of Hodge type inequalities contained in [Laz04, Corollary 1.6.3]; see also [Dol11, Subsection 1.4].

**Fact 1.4.** *Let  $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$  be a Cremona transformation. The following hold:*

- $1 \leq \deg_{i+j}(\varphi) \leq \deg_i(\varphi) \deg_j(\varphi)$ , for  $0 \leq i, j \leq n$ , with  $i + j \leq n$ ;
- $\deg_{i-1}(\varphi) \deg_{i+1}(\varphi) \leq \deg_i(\varphi)^2$ , for  $1 \leq i \leq n-1$ .

The next result concerns the computation of the projective degrees. The only non-trivial part is the second equality of (1.2), and this is essentially shown in [CK89, p. 291]. It is a special case of the well-known relationship between the projective degrees of a rational map from  $\mathbb{P}^n$  and the Segre class of the base locus in  $\mathbb{P}^n$  (see [Ful84, Proposition 4.4] and [Dol11, Subsection 2.3]).

**Fact 1.5.** *Let  $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$  be a special Cremona transformation of type  $(\delta_1, \delta_2)$ . Let  $\mathfrak{B}$  be its base locus,  $r = \dim(\mathfrak{B})$ , and let  $s_i(\mathfrak{B}) = s_i(\mathcal{N}_{\mathfrak{B}/\mathbb{P}^n}) H_{\mathfrak{B}}^{r-i}$  be the degree of the  $i$ -th Segre class of the normal bundle of  $\mathfrak{B}$ . Then the following hold for  $k = 0, \dots, n$ :*

$$(1.1) \quad \deg_0(\varphi) = \deg_0(\varphi^{-1}) = 1, \quad \deg_1(\varphi) = \delta_1, \quad \deg_1(\varphi^{-1}) = \delta_2;$$

$$(1.2) \quad \begin{aligned} \deg_k(\varphi^{-1}) &= \deg_{n-k}(\varphi) \\ &= \delta_1^{n-k} - \binom{n-k}{r-k} \delta_1^{r-k} \deg(\mathfrak{B}) - \sum_{i=k}^{r-1} \binom{n-k}{i-k} \delta_1^{i-k} s_{r-i}(\mathfrak{B}). \end{aligned}$$

**1.3. Some general constraints on special Cremona transformations.** Let  $\varphi$  be a special Cremona transformation of type  $(\delta_1, \delta_2)$  of  $\mathbb{P}^n$ , and let  $\mathfrak{B}$  (resp.  $\mathfrak{B}'$ ) denote the base locus of  $\varphi$  (resp. of  $\varphi^{-1}$ ). In [ESB89, Lemma 2.4] (see also [CK89]) the following formulas are obtained:

$$(1.3) \quad n = \frac{(\delta_1 \delta_2 - 1) \dim \mathfrak{B} + (\delta_1 + 1) \delta_2 - 2}{(\delta_1 - 1) \delta_2} = \frac{(\delta_1 \delta_2 - 1) \dim \mathfrak{B}' + (\delta_2 + 1) \delta_1 - 2}{(\delta_2 - 1) \delta_1},$$

and one has  $2 \leq \delta_1, \delta_2 \leq n$  and  $1 \leq \dim \mathfrak{B}, \dim \mathfrak{B}' \leq n-2$ . When  $n \geq 6$ , one has  $\delta_1 \leq n-1$  and  $\dim \mathfrak{B} \leq n-3$ , see [ESB89, Theorem 3.2].

Let  $k_0 = k_0(\varphi) = (n - \dim \mathfrak{B})\delta_1 - (n+1)$ . A simple application of the Kawamata-Viehweg vanishing theorem applied on the blowing-up of  $\mathbb{P}^n$  along the base locus of  $\varphi$  gives the following (see e.g. [Sta12, Lemma 4.3], [Rus09, Proposition 4.2(ii)], or [Ver01, Corollary 2.12]):

$$(1.4) \quad h^i(\mathbb{P}^n, \mathcal{I}_{\mathfrak{B}, \mathbb{P}^n}(k)) = 0, \quad \text{for } i \geq 1 \text{ provided } k \geq k_0.$$

As one has  $h^0(\mathbb{P}^n, \mathcal{I}_{\mathfrak{B}, \mathbb{P}^n}(\delta_1 - 1)) = 0$  and  $h^0(\mathbb{P}^n, \mathcal{I}_{\mathfrak{B}, \mathbb{P}^n}(\delta_1)) = n+1$ , by (1.4) one always gets at least two conditions for the Hilbert polynomial of  $\mathfrak{B}$ . Note also that for  $k \geq k_0 + 1$ , (1.4) is a consequence of the main result in [BEL91].

**1.4. Four-secants of surfaces in  $\mathbb{P}^5$ .** Recall the formula, due to P. Le Barz, calculating the number of 4-secant lines of a surface  $S \subset \mathbb{P}^5$ .

**Fact 1.6** ([LB80], see also [LB81]). *Let  $S \subset \mathbb{P}^5$  be a smooth surface that contains at most finitely many lines and put  $\kappa = K_S H_S$ ,  $\zeta = K_S^2$ ,  $\theta = c_2(\mathcal{T}_S)$ , and  $\lambda = H_S^2$ . Then the number of 4-secant lines of  $S$ , if finite, is given by*

$$\begin{aligned} & 24^{-1} (3\lambda^4 - 36\kappa\lambda^2 - 6\zeta\lambda^2 + 6\theta\lambda^2 - 90\lambda^3 + 78\kappa^2 \\ & \quad + 30\kappa\zeta + 3\zeta^2 - 30\kappa\theta - 6\zeta\theta + 3\theta^2 + 612\kappa\lambda + 116\zeta\lambda - 100\theta\lambda \\ & \quad + 855\lambda^2 - 1980\kappa - 510\zeta + 294\theta - 2466\lambda) - \sum_{\substack{l \text{ line} \\ l \subset S}} \binom{5+l^2}{4}. \end{aligned}$$

## 2. PROOF OF THEOREM 0.5

Let  $\varphi$  be a special cubic Cremona transformation whose base locus has dimension 3. By (1.3) this is the same as saying that  $\varphi$  is a special cubic Cremona transformation of  $\mathbb{P}^6$ . Then  $\varphi$  is of type  $(3, 5)$ , and the base locus  $\mathfrak{B} \subset \mathbb{P}^6$  is a smooth irreducible threefold cut out by 7 cubic hypersurfaces.

**2.1. Numerical invariants of the transformation.** In this subsection, we compute the numerical invariants of  $\mathfrak{B}$  and  $\varphi$  as functions of  $\lambda$  and  $g$ , where  $\lambda$  and  $g$  denote, respectively, the degree and the sectional genus of  $\mathfrak{B}$ .

**Lemma 2.1.** *The following hold:*

- $h^0(\mathbb{P}^6, \mathcal{I}_{\mathfrak{B}, \mathbb{P}^6}(1)) = h^0(\mathbb{P}^6, \mathcal{I}_{\mathfrak{B}, \mathbb{P}^6}(2)) = 0$  and  $h^0(\mathbb{P}^6, \mathcal{I}_{\mathfrak{B}, \mathbb{P}^6}(3)) = 7$ ;
- $h^i(\mathbb{P}^6, \mathcal{I}_{\mathfrak{B}, \mathbb{P}^6}(j)) = 0$ , for every  $i \geq 1$  and  $j \geq 2$ ;
- $h^1(\mathbb{P}^6, \mathcal{I}_{\mathfrak{B}, \mathbb{P}^6}(1)) = 0$ , i.e.  $\mathfrak{B}$  is linearly normal.

*Proof.* The first statement is contained in [ESB89, Proposition 2.5], and the second follows by (1.4). If  $\mathfrak{B} \subset \mathbb{P}^6$  is not linearly normal, then there exists a smooth nondegenerate threefold  $X \subset \mathbb{P}^7$  with  $\text{Sec}(X) \neq \mathbb{P}^7$  and such that  $X$  is projected isomorphically onto  $\mathfrak{B}$ . But such  $X$ 's are completely classified in [Fuj82], and none of them may be possible in our case.  $\square$



**Lemma 2.2.** *The Hilbert polynomial of  $\mathfrak{B}$  is*

$$\begin{aligned} \chi(\mathcal{O}_{\mathfrak{B}}(tH_{\mathfrak{B}})) &= \lambda \binom{t+3}{3} + (-\lambda - g + 1) \binom{t+2}{2} \\ &\quad + (-6\lambda + 4g + 45)(t+1) + (14\lambda - 6g - 113). \end{aligned}$$

*Proof.* From Lemma 2.1 and the exact sequence  $0 \rightarrow \mathcal{I}_{\mathfrak{B}, \mathbb{P}^6} \rightarrow \mathcal{O}_{\mathbb{P}^6} \rightarrow \mathcal{O}_{\mathfrak{B}} \rightarrow 0$ , we deduce two conditions for the Hilbert polynomial of  $\mathfrak{B}$ . That is

$$\begin{aligned} \chi(\mathcal{O}_{\mathfrak{B}}(3)) &= h^0(\mathcal{O}_{\mathfrak{B}}(3)) = h^0(\mathcal{O}_{\mathbb{P}^6}(3)) - h^0(\mathcal{I}_{\mathfrak{B}, \mathbb{P}^6}(3)) = 77, \\ \chi(\mathcal{O}_{\mathfrak{B}}(2)) &= h^0(\mathcal{O}_{\mathfrak{B}}(2)) = h^0(\mathcal{O}_{\mathbb{P}^6}(2)) = 28. \end{aligned} \quad \square$$

**Lemma 2.3.** *Let  $K_{\mathfrak{B}}$  and  $H_{\mathfrak{B}}$  be, respectively, a canonical divisor and a hyperplane section divisor of  $\mathfrak{B}$ . For  $i = 1, 2, 3$ , let  $c_i = c_i(\mathcal{T}_{\mathfrak{B}})H_{\mathfrak{B}}^{3-i}$  (resp.  $c_i = c_i(\mathcal{N}_{\mathfrak{B}, \mathbb{P}^6})H_{\mathfrak{B}}^{3-i}$ ) be the degree of the  $i$ -th Chern class of the tangent (resp. normal) bundle of  $\mathfrak{B}$ , and let  $s_i = s_i(\mathcal{N}_{\mathfrak{B}, \mathbb{P}^6})H_{\mathfrak{B}}^{3-i}$  (resp.  $s_i = s_i(\mathcal{T}_{\mathfrak{B}})H_{\mathfrak{B}}^{3-i}$ ) be the degree of the  $i$ -th Segre class of the normal (resp. tangent) bundle of  $\mathfrak{B}$ . Also, let  $S \subset \mathbb{P}^5$  denote a smooth hyperplane section of  $\mathfrak{B}$ . Then, with this notation, we have:*

- (i)  $c_1 = 2\lambda - 2g + 2$ ,  $c_2 = -29\lambda + 16g + 222$ ,  $c_3 = 230\lambda - 102g - 1788$ ;
- (ii)  $s_1 = -5\lambda - 2g + 2$ ,  $s_2 = -15\lambda + 30g + 208$ ,  $s_3 = 405\lambda - 270g - 3286$ ;
- (iii)  $c_1 = 5\lambda + 2g - 2$ ,  $c_2 = -3\lambda + 12g + 100$ ,  $c_3 = \lambda^2$ ;
- (iv)  $s_1 = -2\lambda + 2g - 2$ ,  $s_2 = -10\lambda - 2g + 114$ ,  $s_3 = \lambda^2 + 77\lambda - 28g - 756$ ;
- (v)  $K_{\mathfrak{B}}H_{\mathfrak{B}}^2 = -2\lambda + 2g - 2$ ,  $K_{\mathfrak{B}}^2H_{\mathfrak{B}} = -39\lambda + 14g + 336$ ,  $K_{\mathfrak{B}}^3 = \lambda^2 - 77\lambda + 14g + 672$ ;
- (vi)  $K_S H_S = -\lambda + 2g - 2$ ,  $K_S^2 = -42\lambda + 18g + 332$ ,  $c_2(\mathcal{T}_S) = -30\lambda + 18g + 220$ .

*Proof.* For every smooth threefold  $\mathfrak{B}$ , by Hirzebruch-Riemann-Roch formula (see [Har77, App. A, Exercise 6.7]) and using that  $s(\mathcal{T}_{\mathfrak{B}}) = c(\mathcal{T}_{\mathfrak{B}})^{-1}$  we deduce:

$$\begin{aligned} (2.1) \quad s_1(\mathcal{T}_{\mathfrak{B}})H_{\mathfrak{B}}^2 &= -c_1(\mathcal{T}_{\mathfrak{B}})H_{\mathfrak{B}}^2 = K_{\mathfrak{B}}H_{\mathfrak{B}}^2, \\ (2.2) \quad s_2(\mathcal{T}_{\mathfrak{B}})H_{\mathfrak{B}} &= c_1(\mathcal{T}_{\mathfrak{B}})^2H_{\mathfrak{B}} - c_2(\mathcal{T}_{\mathfrak{B}})H_{\mathfrak{B}} = K_{\mathfrak{B}}^2H_{\mathfrak{B}} - c_2(\mathcal{T}_{\mathfrak{B}})H_{\mathfrak{B}}, \\ (2.3) \quad s_3(\mathcal{T}_{\mathfrak{B}}) &= -c_1(\mathcal{T}_{\mathfrak{B}})^3 + 2c_1(\mathcal{T}_{\mathfrak{B}})c_2(\mathcal{T}_{\mathfrak{B}}) - c_3(\mathcal{T}_{\mathfrak{B}}) \\ &= K_{\mathfrak{B}}^3 + 48\chi(\mathcal{O}_{\mathfrak{B}}) - c_3(\mathcal{T}_{\mathfrak{B}}), \\ (2.4) \quad K_{\mathfrak{B}}H_{\mathfrak{B}}^2 &= 2(g - 1 - H_{\mathfrak{B}}^3), \\ (2.5) \quad K_{\mathfrak{B}}^2H_{\mathfrak{B}} &= 12(\chi(\mathcal{O}_{\mathfrak{B}}(H_{\mathfrak{B}})) - \chi(\mathcal{O}_{\mathfrak{B}})) - 2H_{\mathfrak{B}}^3 + 3K_{\mathfrak{B}}H_{\mathfrak{B}}^2 - c_2(\mathcal{T}_{\mathfrak{B}})H_{\mathfrak{B}}. \end{aligned}$$

Using that  $\mathfrak{B}$  is embedded in  $\mathbb{P}^6$ , we also obtain the relation (see [LS86, p. 543]):

$$(2.6) \quad K_{\mathfrak{B}}^3 = c_3(\mathcal{T}_{\mathfrak{B}}) + 7c_2(\mathcal{T}_{\mathfrak{B}})H_{\mathfrak{B}} - 48\chi(\mathcal{O}_{\mathfrak{B}}) + (H_{\mathfrak{B}}^3)^2 - 35H_{\mathfrak{B}}^3 - 21K_{\mathfrak{B}}H_{\mathfrak{B}}^2 - 7K_{\mathfrak{B}}^2H_{\mathfrak{B}}.$$

Moreover, from the exact sequence  $0 \rightarrow \mathcal{T}_{\mathfrak{B}} \rightarrow \mathcal{T}_{\mathbb{P}^6}|_{\mathfrak{B}} \rightarrow \mathcal{N}_{\mathfrak{B}, \mathbb{P}^6} \rightarrow 0$  and since  $s(\mathcal{N}_{\mathfrak{B}, \mathbb{P}^6}) = c(\mathcal{N}_{\mathfrak{B}, \mathbb{P}^6})^{-1}$  and  $s(\mathcal{T}_{\mathfrak{B}}) = c(\mathcal{T}_{\mathfrak{B}})^{-1}$ , we deduce:

$$\begin{aligned} (2.7) \quad c_1(\mathcal{T}_{\mathfrak{B}})H_{\mathfrak{B}}^2 &= 7H_{\mathfrak{B}}^3 + s_1(\mathcal{N}_{\mathfrak{B}, \mathbb{P}^6})H_{\mathfrak{B}}^2, \\ (2.8) \quad c_2(\mathcal{T}_{\mathfrak{B}})H_{\mathfrak{B}} &= 21H_{\mathfrak{B}}^3 + 7s_1(\mathcal{N}_{\mathfrak{B}, \mathbb{P}^6})H_{\mathfrak{B}}^2 + s_2(\mathcal{N}_{\mathfrak{B}, \mathbb{P}^6})H_{\mathfrak{B}}, \\ (2.9) \quad c_3(\mathcal{T}_{\mathfrak{B}}) &= 35H_{\mathfrak{B}}^3 + 21s_1(\mathcal{N}_{\mathfrak{B}, \mathbb{P}^6})H_{\mathfrak{B}}^2 + 7s_2(\mathcal{N}_{\mathfrak{B}, \mathbb{P}^6})H_{\mathfrak{B}} + s_3(\mathcal{N}_{\mathfrak{B}, \mathbb{P}^6}); \end{aligned}$$

$$(2.10) \quad c_1(\mathcal{N}_{\mathfrak{B}, \mathbb{P}^6}) H_{\mathfrak{B}}^2 = 7 H_{\mathfrak{B}}^3 + s_1(\mathcal{T}_{\mathfrak{B}}) H_{\mathfrak{B}}^2,$$

$$(2.11) \quad c_2(\mathcal{N}_{\mathfrak{B}, \mathbb{P}^6}) H_{\mathfrak{B}} = 21 H_{\mathfrak{B}}^3 + 7 s_1(\mathcal{T}_{\mathfrak{B}}) H_{\mathfrak{B}}^2 + s_2(\mathcal{T}_{\mathfrak{B}}) H_{\mathfrak{B}},$$

$$(2.12) \quad c_3(\mathcal{N}_{\mathfrak{B}, \mathbb{P}^6}) = 35 H_{\mathfrak{B}}^3 + 21 s_1(\mathcal{T}_{\mathfrak{B}}) H_{\mathfrak{B}}^2 + 7 s_2(\mathcal{T}_{\mathfrak{B}}) H_{\mathfrak{B}} + s_3(\mathcal{T}_{\mathfrak{B}}).$$

Now, using that  $\mathfrak{B}$  is the base locus of a Cremona transformation of  $\mathbb{P}^6$  of type  $(3, 5)$ , from Fact 1.5 we obtain:

$$(2.13) \quad 1 = \deg_6(\varphi) = -540 H_{\mathfrak{B}}^3 - 135 s_1(\mathcal{N}_{\mathfrak{B}, \mathbb{P}^6}) H_{\mathfrak{B}}^2 - 18 s_2(\mathcal{N}_{\mathfrak{B}, \mathbb{P}^6}) H_{\mathfrak{B}} - s_3(\mathcal{N}_{\mathfrak{B}, \mathbb{P}^6}) + 729,$$

$$(2.14) \quad 5 = \deg_5(\varphi) = -90 H_{\mathfrak{B}}^3 - 15 s_1(\mathcal{N}_{\mathfrak{B}, \mathbb{P}^6}) H_{\mathfrak{B}}^2 - s_2(\mathcal{N}_{\mathfrak{B}, \mathbb{P}^6}) H_{\mathfrak{B}} + 243.$$

Finally, for every smooth threefold  $\mathfrak{B}$ , if  $S$  is a smooth hyperplane section, from the exact sequence  $0 \rightarrow \mathcal{T}_S \rightarrow \mathcal{T}_{\mathfrak{B}}|_S \rightarrow \mathcal{O}_S(H_S) \rightarrow 0$ , we get

$$(2.15) \quad c_2(\mathcal{T}_S) = c_2(\mathcal{T}_{\mathfrak{B}}) H_{\mathfrak{B}} + K_S H_S = c_2(\mathcal{T}_{\mathfrak{B}}) H_{\mathfrak{B}} + K_{\mathfrak{B}} H_{\mathfrak{B}}^2 + H_{\mathfrak{B}}^3,$$

and we also have

$$(2.16) \quad K_S^2 = 12 \chi(\mathcal{O}_S) - c_2(\mathcal{T}_S) = 12 (\chi(\mathcal{O}_{\mathfrak{B}}) - \chi(\mathcal{O}_{\mathfrak{B}}(-H_{\mathfrak{B}}))) - c_2(\mathcal{T}_S).$$

Now, using also Lemma 2.2, the proof is reduced to solving a system of linear equations with coefficients rational functions of  $\lambda$  and  $g$ .  $\square$

**Lemma 2.4.** *The projective degrees of  $\varphi$  are as follows:  $\deg_0(\varphi) = 1$ ,  $\deg_1(\varphi) = 3$ ,  $\deg_2(\varphi) = 9$ ,  $\deg_3(\varphi) = 27 - \lambda$ ,  $\deg_4(\varphi) = -7\lambda + 2g + 79$ ,  $\deg_5(\varphi) = 5$ ,  $\deg_6(\varphi) = 1$ .*

*Proof.* It follows directly from Lemma 2.3(ii) and Fact 1.5.  $\square$

**2.2. Reducing to a short list of not excluded cases.** In this subsection we prove the following:

**Lemma 2.5.** *There are 33 not excluded pairs  $(\lambda, g)$ . These are the following:  $(8, 1)$ ,  $(9, 3)$ ,  $(9, 4)$ ,  $(10, 5)$ ,  $(10, 6)$ ,  $(11, 7)$ ,  $(11, 8)$ ,  $(11, 9)$ ,  $(12, 9)$ ,  $(12, 10)$ ,  $(12, 11)$ ,  $(12, 12)$ ,  $(13, 12)$ ,  $(13, 13)$ ,  $(13, 14)$ ,  $(13, 15)$ ,  $(14, 14)$ ,  $(14, 15)$ ,  $(14, 16)$ ,  $(14, 17)$ ,  $(14, 18)$ ,  $(15, 17)$ ,  $(15, 18)$ ,  $(15, 19)$ ,  $(15, 20)$ ,  $(15, 21)$ ,  $(16, 21)$ ,  $(16, 22)$ ,  $(16, 23)$ ,  $(17, 24)$ ,  $(17, 25)$ ,  $(18, 27)$ ,  $(18, 28)$ .*

*Proof.* Firstly, we note that there are a finite number of not excluded pairs  $(\lambda, g)$ . Indeed, since  $\mathfrak{B}$  is a smooth nondegenerate scheme cut out by cubics and of codimension 3, we have  $3 \leq \lambda \leq 27$ . Further, we have  $g \geq 0$  and the well-known Castelnuovo's bound (see e.g. [BB05, p. 4]) provides an upper bound for  $g$  as a function of  $\lambda$ . This rough argument leaves 889 pairs  $(\lambda, g)$ .

Now, from [LS86, Theorem 0.7.3], for any smooth threefold  $(\mathfrak{B}, H_{\mathfrak{B}})$ , denoting by  $S$  a smooth hyperplane section, one has the inequalities:

$$(2.17) \quad K_{\mathfrak{B}}^3 + 6 K_{\mathfrak{B}}^2 H_{\mathfrak{B}} + 15 K_{\mathfrak{B}} H_{\mathfrak{B}}^2 + 20 H_{\mathfrak{B}}^3 - c_3(\mathcal{T}_{\mathfrak{B}}) + 48 \chi(\mathcal{O}_{\mathfrak{B}}) - 6 c_2(\mathcal{T}_{\mathfrak{B}}) H_{\mathfrak{B}} \geq 0,$$

$$(2.18) \quad K_S^2 + 4 K_S H_S + 6 H_S^2 \geq c_2(\mathcal{T}_S),$$

$$(2.19) \quad 2 c_2(\mathcal{T}_S) \geq c_3(\mathcal{T}_{\mathfrak{B}}) - 2g + 2,$$

$$(2.20) \quad -24 \chi(\mathcal{O}_{\mathfrak{B}}) + 3 K_{\mathfrak{B}}^2 H_{\mathfrak{B}} + 15 K_{\mathfrak{B}} H_{\mathfrak{B}}^2 + 2 c_2(\mathcal{T}_{\mathfrak{B}}) H_{\mathfrak{B}} + 20 H_{\mathfrak{B}}^3 + c_3(\mathcal{T}_{\mathfrak{B}}) \geq 0.$$

So, applying Lemmas 2.2 and 2.3, respectively, we obtain:

$$(2.21) \quad \begin{aligned} g &\leq (\lambda^2 + 7\lambda - 102)/10, & g &\geq (5\lambda - 52)/4, \\ g &\geq (145\lambda - 1113)/70, & g &\geq (147\lambda - 1242)/74. \end{aligned}$$

In particular, the first inequality provides a refinement of the Castelnuovo's bound. All these inequalities restrict the set of not excluded pairs  $(\lambda, g)$  to a set of 312 elements.



Finally, using Lemma 2.4, we apply all the inequalities in Fact 1.4. This turns out to be equivalent to applying just the following:

$$(2.22) \quad \begin{aligned} \deg_1(\varphi) \deg_3(\varphi) &\geq \deg_4(\varphi), & \deg_1(\varphi) \deg_4(\varphi) &\geq \deg_5(\varphi), \\ \deg_3(\varphi)^2 &\geq \deg_2(\varphi) \deg_4(\varphi), & \deg_4(\varphi)^2 &\geq \deg_3(\varphi) \deg_5(\varphi), \\ \deg_5(\varphi)^2 &\geq \deg_4(\varphi) \deg_6(\varphi). \end{aligned}$$

Which respectively become:

$$(2.23) \quad \begin{aligned} 4\lambda - 2g + 2 &\geq 0, & -21\lambda + 6g + 232 &\geq 0, \\ \lambda^2 + 9\lambda - 18g + 18 &\geq 0, & 49\lambda^2 - 28\lambda g + 4g^2 - 1101\lambda + 316g + 6106 &\geq 0, \\ 7\lambda - 2g - 54 &\geq 0. \end{aligned}$$

This restricts the set of pairs  $(\lambda, g)$  to the set of 33 elements as stated.  $\square$

**2.3. Applying adjunction theoretic results.** The following result, thanks to Fact 1.1, say us that  $(\mathfrak{B}, H_{\mathfrak{B}})$  admits a unique minimal reduction.

**Lemma 2.6.** *The adjunction map  $\Phi = \Phi_{|K_{\mathfrak{B}} + 2H_{\mathfrak{B}}|}$  is defined and has image of dimension 3. In other words,  $\mathfrak{B} \subset \mathbb{P}^6$  is not a del Pezzo variety, and it is different from a scroll over either a curve or a surface, as well as from a quadric fibration over a curve.*

*Proof.* If  $\mathfrak{B}$  is a scroll over a curve, we must have  $(K_{\mathfrak{B}} + 3H_{\mathfrak{B}})^3 = 0$ ; while if  $\mathfrak{B}$  is a scroll over a surface, or a quadric fibration over a curve, or a del Pezzo variety, then we must have  $(K_{\mathfrak{B}} + 2H_{\mathfrak{B}})^3 = 0$ . From Lemma 2.3(v), these two relations, respectively, become  $\lambda^2 - 455\lambda + 194g + 3642 = 0$  and  $\lambda^2 - 327\lambda + 122g + 2664 = 0$ . But there is no pair  $(\lambda, g)$ , not excluded by Lemma 2.5, that satisfies any of them.  $\square$

Let  $(\mathfrak{X}, H_{\mathfrak{X}})$  denote the minimal reduction of  $(\mathfrak{B}, H_{\mathfrak{B}})$  and let  $v = v(\mathfrak{B}, H_{\mathfrak{B}})$  be the number of exceptional divisors on  $(\mathfrak{B}, H_{\mathfrak{B}})$ . In the following result, we compute the numerical invariants of  $\mathfrak{X}$  as functions of  $\lambda$ ,  $g$ , and  $v$ .

**Lemma 2.7.** *We have*

- (i)  $\chi(\mathcal{O}_{\mathfrak{X}}(tH_{\mathfrak{X}})) = (\lambda + v) \binom{t+3}{3} + (-\lambda - g - v + 1) \binom{t+2}{2} + (-6\lambda + 4g + 45)(t+1) + (14\lambda - 6g - 113)$ ;
- (ii)  $K_{\mathfrak{X}}H_{\mathfrak{X}}^2 = -2d + 2g - 2v - 2$ ,  $K_{\mathfrak{X}}^2H_{\mathfrak{X}} = -39\lambda + 14g + 4v + 336$ ,  $K_{\mathfrak{X}}^3 = \lambda^2 - 77\lambda + 14g - 8v + 672$ ;
- (iii)  $c_1(\mathcal{T}_{\mathfrak{X}})H_{\mathfrak{X}}^2 = 2\lambda - 2g + 2v + 2$ ,  $c_2(\mathcal{T}_{\mathfrak{X}})H_{\mathfrak{X}} = -29\lambda + 16g + 222$ ,  $c_3(\mathcal{T}_{\mathfrak{X}}) = 230\lambda - 102g - 2v - 1788$ ;
- (iv) for a smooth element  $\mathfrak{P} \in |H_{\mathfrak{X}}|$ ,  $K_{\mathfrak{P}}^2 = -42\lambda + 18g + v + 332$ ,  $c_2(\mathcal{T}_{\mathfrak{P}}) = -30\lambda + 18g - v + 220$ .

*Proof.* Let  $\pi : \mathfrak{B} \rightarrow \mathfrak{X}$  be the map of the blowing-up along the points  $p_1, \dots, p_v$ , and denote by  $E_i = \pi^{-1}(p_i)$  the exceptional divisors. Since  $K_{\mathfrak{B}} \approx \pi^*K_{\mathfrak{X}} + 2\sum_{i=1}^v E_i$  and  $H_{\mathfrak{B}} \approx \pi^*H_{\mathfrak{X}} - \sum_{i=1}^v E_i$ , one obtains  $K_{\mathfrak{X}}H_{\mathfrak{X}}^2 = K_{\mathfrak{B}}H_{\mathfrak{B}}^2 - 2v$ ,  $K_{\mathfrak{X}}^2H_{\mathfrak{X}} = K_{\mathfrak{B}}^2H_{\mathfrak{B}} + 4v$ , and  $K_{\mathfrak{X}}^3 = K_{\mathfrak{B}}^3 - 8v$ . One also has  $c_1(\mathcal{T}_{\mathfrak{X}})H_{\mathfrak{X}}^2 = c_1(\mathcal{T}_{\mathfrak{B}})H_{\mathfrak{B}}^2 + 2v$ , and from [GH78, p. 609] it follows that  $c_2(\mathcal{T}_{\mathfrak{X}})H_{\mathfrak{X}} = c_2(\mathcal{T}_{\mathfrak{B}})H_{\mathfrak{B}}$ . Now, using Hirzebruch-Riemann-Roch formula, one deduces that for every  $t \in \mathbb{Z}$ ,  $\chi(\mathcal{O}_{\mathfrak{X}}(tH_{\mathfrak{X}})) = \chi(\mathcal{O}_{\mathfrak{B}}(tH_{\mathfrak{B}})) + v(t^3 + 3t^2 + 2t)/6$ ; in particular,  $\chi(\mathcal{O}_{\mathfrak{X}}(tH_{\mathfrak{X}})) = \chi(\mathcal{O}_{\mathfrak{B}}(tH_{\mathfrak{B}}))$  for  $t = 0, -1, -2$ . As an application of double point formulas, see [BB05, Lemma 3.3], one

deduces that  $s_3(\mathcal{T}_{\mathfrak{R}}) = -21s_1(\mathcal{T}_{\mathfrak{R}})H_{\mathfrak{R}}^2 - 7s_2(\mathcal{T}_{\mathfrak{R}})H_{\mathfrak{R}} + (H_{\mathfrak{R}}^3 - v)^2 - 35H_{\mathfrak{R}}^3 + 15v$ , and from this and (2.6) one has:

$$\begin{aligned}
 c_3(\mathcal{T}_{\mathfrak{R}}) &= K_{\mathfrak{R}}^3 + 48\chi(\mathcal{O}_{\mathfrak{R}}) - s_3(\mathcal{T}_{\mathfrak{R}}) \\
 &= K_{\mathfrak{R}}^3 + 48\chi(\mathcal{O}_{\mathfrak{R}}) + 21K_{\mathfrak{R}}H_{\mathfrak{R}}^2 + 7(K_{\mathfrak{R}}^2H_{\mathfrak{R}} - c_2(\mathcal{T}_{\mathfrak{R}})H_{\mathfrak{R}}) \\
 &\quad - (H_{\mathfrak{R}}^3 - v)^2 + 35H_{\mathfrak{R}}^3 - 15v \\
 &= -7c_2(\mathcal{T}_{\mathfrak{B}})H_{\mathfrak{B}} + 48\chi(\mathcal{O}_{\mathfrak{B}}) - \lambda^2 + 35\lambda + 21K_{\mathfrak{B}}H_{\mathfrak{B}}^2 \\
 &\quad + 7K_{\mathfrak{B}}^2H_{\mathfrak{B}} + K_{\mathfrak{B}}^3 - 2v \\
 &\text{(by (2.6))} = c_3(\mathcal{T}_{\mathfrak{B}}) - 2v.
 \end{aligned}$$

Finally, part (iv) is obtained through the analogue formulas of (2.15) and (2.16).  $\square$

**Proposition 2.8.** *One of the two following possibilities occurs:*

- (A)  $(\mathfrak{R}, H_{\mathfrak{R}})$  is of log-general type, and  $(\lambda, g, v) \in \{(14, 15, 0), (18, 27, 0)\}$ ;
- (B)  $(\mathfrak{R}, H_{\mathfrak{R}})$  is a conic bundle over a surface, and  $v = \lambda^2 - 199\lambda + 62g + 1674$ .

*Proof.* We apply Fact 1.2 in order to recognize when  $K_{\mathfrak{R}} + H_{\mathfrak{R}}$  is nef and big. One sees easily that  $(\mathfrak{R}, H_{\mathfrak{R}})$  cannot be as in Fact 1.2, cases (I) and (II). Assume that  $(\mathfrak{R}, H_{\mathfrak{R}})$  is as in Fact 1.2, case (III). Then, from Lemma 2.7(ii), we obtain the two conditions:

$$(2.24) \quad 0 = (2K_{\mathfrak{R}} + 3H_{\mathfrak{R}})^3 = 8\lambda^2 - 2101\lambda + 724g - v + 17364,$$

$$(2.25) \quad 0 = (2K_{\mathfrak{R}} + 3H_{\mathfrak{R}})^2 H_{\mathfrak{R}} = -171\lambda + 80g + v + 1320.$$

Thus we must have  $v = 8\lambda^2 - 2101\lambda + 724g + 17364 = 171\lambda - 80g - 1320$ , but there are no pairs  $(\lambda, g)$ , not excluded by Lemma 2.5, that satisfy it. Similarly, if  $(\mathfrak{R}, H_{\mathfrak{R}})$  is as in Fact 1.2, case (V), we deduce that  $v = \lambda^2 - 199\lambda + 62g + 1674 = 42\lambda - 18g - 332$ , which is again impossible. Now, assume that  $(\mathfrak{R}, H_{\mathfrak{R}})$  is as in Fact 1.2, case (IV). The three conditions  $K_{\mathfrak{R}}^3 + H_{\mathfrak{R}}^3 = K_{\mathfrak{R}}^2H_{\mathfrak{R}} - H_{\mathfrak{R}}^3 = K_{\mathfrak{R}}H_{\mathfrak{R}}^2 + H_{\mathfrak{R}}^3 = 0$  become  $\lambda^2 - 76\lambda + 14g - 7v + 672 = -40\lambda + 14g + 3v + 336 = -\lambda + 2g - v - 2 = 0$ , but there is no 3-tuple of integers that satisfies them. Finally, if  $(\mathfrak{R}, H_{\mathfrak{R}})$  is as in Fact 1.2, case (VI), i.e. a conic bundle over a surface, we then deduce the relation:  $0 = (K_{\mathfrak{R}} + H_{\mathfrak{R}})^3 = \lambda^2 - 199\lambda + 62g - v + 1674$ .

Now, if  $K_{\mathfrak{R}} + H_{\mathfrak{R}}$  is nef and big, using again Lemma 2.7, we can apply all the inequalities contained in Fact 1.3. For example, the obvious inequality  $d_1 \geq 1$  translates to  $v \leq -\lambda + 2g - 3$ , and in particular we see that only finitely many triples  $(\lambda, g, v)$  are not excluded (more precisely, there are 478 not excluded triples). If now one applies the inequality  $d_1^2 - d_2d_0 \geq 0$ , which translates to  $43\lambda^2 - 22\lambda g + 4g^2 + 43\lambda v - 22gv - 328\lambda - 8g - 328v + 4 \geq 0$ , then only 18 triples are not excluded. Among these, only  $(14, 15, 0)$  and  $(18, 27, 0)$  are not excluded by  $d_2 = -42\lambda + 18g + v + 332 \geq 1$  and  $d_3 = \lambda^2 - 199\lambda + 62g - v + 1674 \geq 1$  (and not excluded by any other inequality stated in Fact 1.3).  $\square$

**2.4. Applying Le Barz's formula.** We can apply Fact 1.6 to the general hyperplane section  $S \subset \mathbb{P}^5$  of  $\mathfrak{B} \subset \mathbb{P}^6$ , by taking into account the following:

- $S \subset \mathbb{P}^5$  can contain at most finitely many lines, and all them have self-intersection  $\leq -1$ ;

- the exceptional divisors of  $\mathfrak{B}$  correspond to the lines  $l \subset S$  with self-intersection  $l^2 = -1$ ; and so we have

$$(2.26) \quad \sum_{\substack{l \text{ line} \\ l \subset S}} \binom{5+l^2}{4} = v + \sum_{\substack{l \text{ line}, l \subset S \\ l^2 \leq -6}} \binom{5+l^2}{4} \geq v;$$

- since  $S \subset \mathbb{P}^5$  is cut out by cubics, it cannot have 4-secant lines.

Thus, using (2.26) and Lemma 2.3(vi), from Fact 1.6 we deduce the following inequality:

$$(2.27) \quad \begin{aligned} v \leq & (1/8)\lambda^4 + (3/4)\lambda^3 - 3\lambda^2g - (453/8)\lambda^2 + 20\lambda g + 13g^2 \\ & + (2835/4)\lambda - 73g - 2894. \end{aligned}$$

Now, from Proposition 2.8 and the inequality (2.27), one deduces that there are only the two following possibilities:

- (A')  $(\lambda, g, v) = (14, 15, 0)$  and  $(\mathfrak{B}, H_{\mathfrak{B}}) = (\mathfrak{R}, H_{\mathfrak{R}})$  is of log-general type;
- (B')  $(\lambda, g, v) = (13, 12, 0)$  and  $(\mathfrak{B}, H_{\mathfrak{B}}) = (\mathfrak{R}, H_{\mathfrak{R}})$  has the structure of a conic bundle over a polarized surface  $(Y, H_Y)$ .

**2.5. Conclusion of the proof.** In case (B') above, (by Lemma 2.7) we get  $h^0(K_{\mathfrak{R}} + H_{\mathfrak{R}}) = h^3(-H_{\mathfrak{R}}) = -\chi(\mathcal{O}_{\mathfrak{R}}(-H_{\mathfrak{R}})) = 3$  and  $d_2(\mathfrak{R}, H_{\mathfrak{R}}) = 2$ . From this, it follows that  $(Y, H_Y) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ , see [BB05, Proposition 11.5]. Assume now that  $\mathfrak{B}$  is as in case (A'). We first show that  $K = K_{\mathfrak{B}}$  is numerically trivial with the same argument as in [Deb01, (3.8), p. 69]. So, let  $C$  be a fixed irreducible curve on  $\mathfrak{B}$ . A standard argument (see for example *loc. cit.*) and Bertini theorem assures that for  $m$  sufficiently large there exists an irreducible and reduced  $Y \in |mH|$  containing  $C$  ( $H = H_{\mathfrak{B}}$  stands for the hyperplane divisor). From Lemma 2.3(v), we see that  $K|_Y \cdot H|_Y = K \cdot H \cdot (mH) = m(K \cdot H^2) = 0$  and that  $K|_Y^2 = K \cdot K \cdot (mH) = m(K^2 \cdot H) = 0$ . Thus, applying Hodge index theorem on the irreducible surface  $Y$  (see [Kle71, Théorème 7.1]), we deduce that  $K|_Y$  is numerically trivial. It follows that  $K \cdot C = K|_Y \cdot C = 0$ , and hence that  $K$  is numerically trivial. Now, from [Kaw85, Theorem 8.2], we deduce that the Kodaira dimension of  $\mathfrak{B}$  is zero, and consequently, from [LS86, Corollary 1.1.2], we deduce that  $K$  is trivial. Now we conclude that  $\mathfrak{B}$  is Pfaffian from the main result in [Wal96].

### 3. PROOF OF THEOREM 0.6

Let  $\varphi$  be a special Cremona transformation of  $\mathbb{P}^7$ . Denote by  $\mathfrak{B}$  (resp.  $\mathfrak{B}'$ ) the base locus of  $\varphi$  (resp.  $\varphi^{-1}$ ), and by  $(\delta_1, \delta_2)$  the type of  $\varphi$ . From (1.3) and since one has  $\dim \mathfrak{B} \leq 4$ , it follows  $\dim \mathfrak{B} = \dim \mathfrak{B}' = 4$  and  $\delta_1 = \delta_2 = 3$ . Thus the restriction of  $\varphi$  to a general hyperplane  $\mathbb{P}^6 \subset \mathbb{P}^7$  is a cubo-cubic birational transformation of  $\mathbb{P}^6$  into a cubic hypersurface of  $\mathbb{P}^7$  whose base locus is a smooth irreducible threefold  $X \subset \mathbb{P}^6$ , which is a general hyperplane section of  $\mathfrak{B}$ . We now conclude the proof of Theorem 0.6 by adapting the proof of Theorem 0.5 in order to get information on  $X$  and hence on  $\mathfrak{B}$ .

*Proof of Theorem 0.6.* Keep the notation as above. As  $\text{codim } \mathfrak{B} = \text{codim } \mathfrak{B}' \geq 3$  we have  $\deg_i(\varphi) = \deg_{7-i}(\varphi) = 3^i$  for  $i = 0, 1, 2$ , and by (1.2) we can express  $\deg_3(\varphi)$  and  $\deg_4(\varphi)$  as functions of  $\lambda = \deg \mathfrak{B}$  and of the sectional genus  $g$  of  $\mathfrak{B}$ ; namely, we have:

$$(3.1) \quad \begin{aligned} \deg_1(\varphi) &= 3, & \deg_2(\varphi) &= 9, & \deg_3(\varphi) &= -\lambda + 27, \\ \deg_4(\varphi) &= -7\lambda + 2g + 79, & \deg_5(\varphi) &= 9, & \deg_6(\varphi) &= 3. \end{aligned}$$

By (1.4) we deduce that  $\mathfrak{B}$  is projectively normal and its Hilbert polynomial,  $P_{\mathfrak{B}}(t)$ , satisfies the conditions:  $P_{\mathfrak{B}}(3) = 112$ ,  $P_{\mathfrak{B}}(2) = 36$ , and  $P_{\mathfrak{B}}(1) = 8$ , from which it follows that  $P_{\mathfrak{B}}(t)$  has the following expression:

$$\begin{aligned} \lambda \binom{t+4}{4} + (-\lambda - g + 1) \binom{t+3}{3} + (-6\lambda + 4g + 44) \binom{t+2}{2} \\ + (14\lambda - 6g - 110)(t+1) + (-11\lambda + 4g + 92). \end{aligned}$$

From (1.2) and (3.1) one determines the degrees of the Segre classes of the normal bundle of  $\mathfrak{B}$ , and then, from the formulas in the proof of Lemma 2.3, one obtains the numerical invariants of  $X$  as functions of  $\lambda$  and  $g$ . In particular, denoting by  $K_X$  and  $H_X$ , respectively, a canonical divisor and a hyperplane section divisor of  $X$ , and by  $S$  a smooth member of  $|H_X|$ , one has:

$$\begin{aligned} (3.2) \quad K_X H_X^2 &= -2\lambda + 2g - 2, \\ K_X^2 H_X &= -39\lambda + 14g + 328, \\ K_X^3 &= \lambda^2 - 77\lambda + 14g + 646; \\ (3.3) \quad K_S^2 &= -42\lambda + 18g + 324, \quad c_2(\mathcal{T}_S) = -30\lambda + 18g + 216. \end{aligned}$$

Now we proceed as in Lemma 2.5, by applying the inequalities (2.17), (2.18), (2.19) and (2.20) to the threefold  $X$ , and the inequalities (2.22) to the Cremona transformation  $\varphi$ . Then, using also that  $\lambda$  and  $g$  are non-negative integers bounded above, respectively, by 27 and by the Castelnuovo's bound, one can show that there are only 27 not excluded pairs  $(\lambda, g)$ , which are the following: (8, 2), (9, 4), (10, 6), (10, 7), (11, 8), (11, 9), (11, 10), (12, 10), (12, 11), (12, 12), (12, 13), (13, 12), (13, 13), (13, 14), (13, 15), (13, 16), (14, 15), (14, 16), (14, 17), (14, 18), (15, 19), (15, 20), (15, 21), (16, 22), (16, 23), (17, 25), (18, 28). One then sees, using (3.2) and the same argument as in the proof of Lemma 2.6, that  $(X, H_X)$  admits a unique minimal reduction  $(X', H')$ , and we denote by  $\nu$  the number of exceptional divisors on  $(X, H_X)$ . Now, quite similarly as we have deduced the inequality (2.27), from (3.3) and Fact 1.6, we deduce the following:

$$\begin{aligned} (3.4) \quad \nu &\leq (1/8)\lambda^4 + (3/4)\lambda^3 - 3\lambda^2 g - (445/8)\lambda^2 + 20\lambda g + 13g^2 \\ &\quad + (2815/4)\lambda - 83g - 2873. \end{aligned}$$

In particular, we see that only finitely many triples  $(\lambda, g, \nu)$  are not excluded (precisely, there are 859 not excluded triples). We now show that  $(X', H')$  is not of log-general type. Assume by contradiction that  $K' + H'$  is nef and big, where  $K'$  stands for the canonical divisor of  $X'$ . From (3.2) (see also the proof of Lemma 2.7) we can determine the pluridegrees  $d_0, \dots, d_3$  of  $(X', H')$  as functions of  $\lambda$ ,  $g$ , and  $\nu$ :

$$\begin{aligned} (3.5) \quad d_0 &= \lambda + \nu, \quad d_1 = -\lambda + 2g - \nu - 2, \\ d_2 &= -42\lambda + 18g + \nu + 324, \quad d_3 = \lambda^2 - 199\lambda + 62g - \nu + 1624. \end{aligned}$$

Then by Fact 1.3 in particular we deduce:

$$\begin{aligned} d_1 &\geq 1, \quad d_2 \geq 1, \quad d_3 \geq 1, \\ d_1^2 - d_2 d_0 &= 43\lambda^2 - 22\lambda g + 4g^2 + 43\lambda \nu - 22g \nu - 320\lambda - 8g - 320\nu + 4 \geq 0. \end{aligned}$$

But one can see that the set of triples  $(\lambda, g, v)$  satisfying these inequalities, with  $(\lambda, g)$  being one of the 27 not excluded pairs, consists of only one element which is  $(18, 28, 0)$ . So (3.5) becomes:

$$d_0 = 18, \quad d_1 = 36, \quad d_2 = 72, \quad d_3 = 102.$$

This is impossible by [BBS89, Lemma 1.1, (1.1.2)], because the equality  $d_1^2 - d_2d_0 = 0$  should imply  $d_2^2 - d_3d_1 = 0$ , and this is not the case. It follows that  $(X', H')$  is as one of the six cases of Fact 1.2. Cases (I), (II), (III), and (IV) are easily excluded, arguing as at the beginning of the proof of Proposition 2.8. Therefore  $(X', H')$  is as in case either (V) or (VI). Then we must have  $d_3 = (K' + H')^3 = 0$ , which by (3.5) means  $v = \lambda^2 - 199\lambda + 62g + 1624$ . But from this and (3.4) we get that only the triple  $(\lambda, g, v) = (12, 10, 0)$  is not excluded, and in particular (3.5) becomes:  $d_0 = 12, d_1 = 6, d_2 = 0, d_3 = 0$ . By  $d_2 = 0$  we deduce that case (VI) is impossible. Thus we conclude that  $(X', H')$  is as in case (V), i.e. a del Pezzo fibration over a polarized curve  $(C, H_C)$ , and  $(X, H_X) = (X', H')$  is a minimal threefold of degree 12 and sectional genus 10. Moreover, denoting by  $F$  the generic fibre of the fibration, the following facts are known (see [BB05, p. 13]):

$$g(C) = 1 - \chi(\mathcal{O}_X), \quad \deg(H_C) = -\chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-H_X)), \quad \deg(F) = d_1 / \deg(H_C),$$

from which it follows that  $(C, H_C) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$  and  $\deg(F) = 6$ .  $\square$

#### 4. EXAMPLES

**4.1. A threefold of degree 14 and sectional genus 15.** As it was pointed out in [ESB89, Subsection 4.2] (see also [CK91, Example on p. 282]), the Pfaffians of the principal  $6 \times 6$  minors of a general  $7 \times 7$  skew-symmetric matrix of linear forms on  $\mathbb{P}^6$  give a Cremona transformation of  $\mathbb{P}^6$ . Its base locus  $X \subset \mathbb{P}^6$ , i.e. the vanishing locus of the Pfaffians, is a smooth and irreducible threefold of degree 14 and sectional genus 15, with trivial canonical bundle and  $h^1(\mathcal{O}_X) = 0$ . Therefore, examples for case (A) of Theorem 0.5 exist. One can also find explicit equations for such an  $X$  with the help of a computer algebra system as MACAULAY2.

**4.2. A threefold of degree 13 and sectional genus 12.** Here, we construct an example of special Cremona transformation as in case (B) of Theorem 0.5. We first construct a threefold in  $\mathbb{P}^6$  of degree 13 and sectional genus 12, but which turns out to be singular. Let  $C \subset \mathbb{P}^3$  be the isomorphic projection of the quintic rational normal curve together with an embedded point  $p_0$ , and let  $p_1, \dots, p_5$  be 5 general points on a general plane passing through  $p_0$ . Then the homogeneous ideal of the non-reduced scheme  $C \cup \{p_1, \dots, p_5\} \subset \mathbb{P}^3$  is generated by 7 quartics, which define a quarto-quadric birational map  $\mathbb{P}^3 \dashrightarrow \mathbb{P}^6$ . The image of this map is an irreducible singular threefold of degree 13, sectional genus 12, and cut out by 7 cubics. These 7 cubics give a cubo-quintic Cremona transformation of  $\mathbb{P}^6$ . The next step is to deform this Cremona transformation in order to obtain a special Cremona transformation with the same invariants. The idea is to take random restrictions and liftings of the map, but this is easier to do than to explain. In the following MACAULAY2 code, we get the desired example.

Macaulay2, version 1.9.2.1

```
with packages: ConwayPolynomials, Elimination, IntegralClosure,
               LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone
i1 : loadPackage "Cremona";
i2 : K = QQ; P1 = K[u_0, u_1]; P3 = K[t_0..t_3]; P6 = K[x_0..x_6];
i6 : curve = trim kernel map(P1, P3, {u_1^5, u_0^5 - u_0^3 * u_1^2, u_0^4 * u_1, u_0 * u_1^4});
o6 : Ideal of P3
i7 : points = intersect {ideal(t_3, t_2, t_1), ideal(t_3, t_2, t_0 - t_1),
```

```

ideal(t_2-t_3,t_1+t_3,t_0-t_3),ideal(t_2-t_3,t_1+t_3,t_0+t_3),
ideal(t_2-t_3,t_1-t_3,t_0),ideal(t_2-t_3,t_1-t_3,t_0+t_3)};
o7 : Ideal of P3
i8 : time phi = rationalMap kernel(map(P3,P6, gens saturate(points * curve)),3)
-- used 1.77549 seconds
o8 = cubic rational map from PP^6 to PP^6
o8 : RationalMap
i9 : time phi = lift invertBirMap rationalMap(flatten(
(invertBirMap phi)|ideal(x_0-x_1,x_2-x_4)),Dominant=>3)
-- used 104.747 seconds
o9 = cubic rational map from PP^6 to PP^6
o9 : RationalMap
i10 : time phi = invertBirMap lift invertBirMap rationalMap(
flatten(phi|ideal(x_0)),Dominant=>true)
-- used 40.6531 seconds
o10 = cubic rational map from PP^6 to PP^6

```

Furthermore, the package *Cremona* ( $v \geq 3.9.1$ ) provides the method `specialCremonaTransformation` which is able to produce examples of special Cremona transformations according to Table 1. In particular, we can obtain the example above as follows:

```

i11 : time phi = specialCremonaTransformation 12
-- used 0.208033 seconds
o11 = Cremona transformation of PP^6 of type (3,5)
o11 : RationalMap
i12 : time projectiveDegrees phi
-- used 0.000019401 seconds
o12 = {1, 3, 9, 14, 12, 5, 1}
o12 : List

```

Once we have the explicit transformation, the unique non-trivial condition to check is the smoothness of the base locus. This can be done by reducing the characteristic (see [Sta15, Section 2.5]).

*Remark 4.1.* Following [MP97, Example 5.4 and Theorem 4.1], one can construct another smooth threefold in  $\mathbb{P}^6$  of degree 13 and sectional genus 12, but which is not cut out by cubics.

**4.3. A threefold of degree 12 and sectional genus 10.** According to [BT15, Case 6a, p. 260], a threefold in  $\mathbb{P}^6$  of degree 12, sectional genus 10, with a structure of fibration over  $\mathbb{P}^1$ , the fibers of which are sextic del Pezzo surfaces, exists. This threefold is contained in a quadric hypersurface of  $\mathbb{P}^6$ , so that it does not define a cubo-cubic transformation as the one constructed via restriction to a general  $\mathbb{P}^6$  of the transformation in Theorem 0.6.<sup>1</sup>

Moreover, a cubo-cubic Cremona transformation of  $\mathbb{P}^7$  (linearly equivalent to an involution) whose base locus is a fourfold of degree 12 and sectional genus 10 exists. This is the polar map of the dual hypersurface of  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$ ; its base locus consists of the union of three  $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$  intersecting pairwise in a  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$ .

**4.4. Two threefolds of degree 10 and sectional genus 6.** We conclude with two further examples of cubic birational transformations of  $\mathbb{P}^6$ .

---

<sup>1</sup>Note that there also exist smooth curves of degree 6 and genus 3 in  $\mathbb{P}^3$ , contained in a quadric surface, that do not define cubo-cubic transformations as in case (1) of Theorem 0.1.



4.4.1. The  $3 \times 3$  minors of a general  $3 \times 5$  matrix of linear forms on  $\mathbb{P}^6$  give a special cubo-cubic birational transformation of  $\mathbb{P}^6$  into the Grassmannian  $\mathbb{G}(2, 4) \subset \mathbb{P}^9$ . Its base locus  $X \subset \mathbb{P}^6$ , i.e. the vanishing locus of the  $3 \times 3$  minors, is a scroll over  $\mathbb{P}^2$  of degree 10 and sectional genus 6.

4.4.2. The special quadro-cubic birational transformation of  $\mathbb{P}^8$  into a factorial cubic hypersurface of  $\mathbb{P}^9$ , described in [Sta12, Example 5.5], can be extended (using the same method described in [Sta15]) to a quadro-cubic Cremona transformation of  $\mathbb{P}^9$ . The restriction to a general  $\mathbb{P}^6 \subset \mathbb{P}^9$  of the inverse of this Cremona transformation gives us an example of special cubo-quadric birational transformation of  $\mathbb{P}^6$  into a factorial complete intersection of three quadrics in  $\mathbb{P}^9$ . Its base locus is a smooth irreducible threefold of degree 10 and sectional genus 6 that has a structure of scroll over  $\mathbb{P}^2$  with four double points blown up; the existence of such a threefold had been left undecided in [FL97].

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